

SCHRÖDINGER MAPS AS A GAUGE THEORY

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ABSTRACT. We consider energy-critical Schrödinger maps from \mathbb{R}^2 into \mathbb{S}^2 and \mathbb{H}^2 . Viewing such maps with respect to orthonormal frames on the pullback bundle provides a gauge field formulation of the evolution. We show that this gauge field system is the set of Euler-Lagrange equations corresponding to an action, which is seen to include a Chern-Simons term. We also consider harmonic map heat flow and harmonic maps from a related point of view. For the Schrödinger evolution, we derive several conservation laws. These conservation laws give rise to virial identities and suggest a framework for proving Morawetz estimates. We conclude by comparing the gauged Schrödinger map system with the Chern-Simons-Schrödinger system.

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1. INTRODUCTION

Our purpose in this article is to develop a gauge field variational approach for obtaining some particular well-known gauge field systems that alternatively may be derived from an underlying geometric map. We employ our approach in dimension $d = 2$, where the underlying geometric map equations of interest are energy-critical.

One motivation for pursuing such a variational derivation is the desire to lift all salient aspects of the geometric map equations to the gauge field level. This is because in practice the analysis of these equations lends itself to the gauge field formulation, and many state-of-the-art results are obtained by working with gauge field formulations.

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An advantage of the gauge field variational formulation is that it clarifies the role of the underlying symmetries and the application of Noether's theorem. Indeed, we derive several natural conservation laws. Though these relations are gauge invariant, and therefore hold at the level of the map, we believe that the gauge field approach that we adopt here clarifies the important role of the underlying geometry and places us in a better analytic framework for proving estimates. In particular, we anticipate that our approach will be useful in establishing virial identities and various Morawetz estimates, and we touch upon these topics briefly.

The thrust of this article is toward developing a framework for studying the energy-critical Schrödinger map system with target \mathbb{S}^2 or target \mathbb{H}^2 . It will be seen to enjoy many characteristics analogous to those enjoyed by the Chern-Simons-Schrödinger system. At the same time, there are important differences, an interesting consequence of which is the relative ease with which finite-time blow-up solutions may be constructed for the Chern-Simons-Schrödinger system as compared to the effort required for Schrödinger maps.

Nearing the completion of this article it was discovered that aspects of our approach to Schrödinger maps have precedent in the physics literature, see [32], where an action at the gauge field level is introduced and conservation laws are derived for the case where the target is \mathbb{S}^2 . The action that we introduce, however, is slightly different, and this difference is key to obtaining all constraints internally from the variation rather than having to impose them from without as is done in [32]. The use of “reduced fields” $\psi_{\pm} := \psi_1 \pm i\psi_2$, which in the mathematics literature have recently been put to good use in studying Schrödinger maps, e.g. see [1, 2], also has precedent in [32]. As our approach in this article is geometric and our focus is toward establishing virial identities and Morawetz estimates, we strongly recommend that [32] be consulted for physical interpretations.

1.1. Geometric map equations. Suppose we have $\phi : \mathbb{R}^d \rightarrow M$, where \mathbb{R}^d is Euclidean space, M is a Riemannian manifold with metric h , and ϕ is a smooth map. Consider the Lagrangian

$$\frac{1}{2} \int_{\mathbb{R}^d} \langle \partial_j \phi, \partial_j \phi \rangle_{h(\phi(x))} dx \quad (1)$$

where here and throughout we sum repeated Roman indices over all spatial variables. The associated Euler-Lagrange equation is

$$(\phi^* \nabla)_j \partial_j \phi = 0 \quad (2)$$

the solutions of which are called *harmonic maps*. Here ∇ denotes the Levi-Civita connection on M and $\phi^* \nabla$ denotes the pullback of this connection to \mathbb{R}^d . The downward gradient flow associated to (1) generates the *harmonic*

map heat flow equation

$$\partial_t \phi = (\phi^* \nabla)_j \partial_j \phi \quad (3)$$

If the target manifold M happens to be Kähler, so that it carries a complex structure J , then we would like to introduce in the action a Schrödinger-like term. As this term ought only to carry one derivative, the natural pairing is with a 1-form. There can, however, be topological obstructions to global nonvanishing 1-forms, such as is the case for \mathbb{S}^2 . To handle the target \mathbb{S}^2 one may first stereographically project to \mathbb{C} and then on that level write down a suitable action [32, 18], though this procedure does not genuinely circumvent the fundamental topological issue. In any case, on the level of maps we are led to the *Schrödinger map* equation

$$\partial_t \phi = J(\phi)(\phi^* \nabla)_j \partial_j \phi \quad (4)$$

From the physical point of view, equation (4) arises in ferromagnetism as the Heisenberg model for a ferromagnetic spin system; it describes the classical spin [27, 37, 32, 6, 36].

Solutions of the above equations all enjoy scale invariance. Solutions of (2) are preserved by the scaling

$$\phi(x) \mapsto \phi(\lambda x) \quad \lambda > 0$$

and solutions of (3) and (4) are preserved by the scaling

$$\phi(t, x) \mapsto \phi(\lambda^2 t, \lambda x) \quad \lambda > 0$$

For each of these equations, the natural energy is also given by (1):

$$E(\phi) := \frac{1}{2} \int_{\mathbb{R}^d} \langle \partial_j \phi, \partial_j \phi \rangle_{h(\phi(x))} dx$$

Energy is formally conserved by (4), and as noted the flow of (3) is the downward gradient flow associated to the energy. The energy of rescaled solutions scales according to

$$E(\phi(x)) = \lambda^{2-d} E(\phi(\lambda x))$$

In dimension $d = 2$, both the equation and the energy are preserved by rescalings. This is called the energy-critical setting, and from now on we assume $d = 2$.

The literature on harmonic maps and harmonic map heat flow is vast and we make no attempt to survey it here, other than to point readers to the works [28, 42, 15, 20] and the references therein. The wellposedness theory of Schrödinger maps is not nearly as developed but recently much progress has been made. For the critical small data global wellposedness theory, see [3]. See also [6, 11, 36, 19, 35] and the references therein.

1.2. Gauges. One theme unifying the study of equations (2)—(4) is the use of *gauges* or *moving frames*: for each point in the domain, e.g. each $(t, x) \in I \times \mathbb{R}^2$ for cases (3) and (4), we choose an orthonormal basis of $TM_{\phi(t,x)}$. Frames have been used extensively in studying harmonic maps [20], and their use in the setting of Schrödinger maps in proving wellposedness was initiated in [6]. Our notation and perspective follow closely that in [44, Chapter 6]. In the energy-critical case with a surface as the target, we have one degree of freedom in our choice of orthonormal frame for each (t, x) . For maps from \mathbb{R}^2 into $M \in \{\mathbb{S}^2, \mathbb{H}^2\}$ evolving on some time interval I , a gauge choice may be represented by the diagram

$$\begin{array}{ccccc} \mathbb{R}^2 \times \mathbb{R}^2 & \xrightarrow{e} & \phi^* TM & \longrightarrow & TM \\ \uparrow \psi_\alpha & & \uparrow \partial_\alpha \phi & & \downarrow \pi \\ I \times \mathbb{R}^2 & \xrightarrow{id} & I \times \mathbb{R}^2 & \xrightarrow{\phi} & M \end{array}$$

Here $\psi_\alpha = e^* \partial_\alpha \phi$ denotes the vector $\partial_\alpha \phi$ written with respect to the choice of orthonormal frame, represented by the map $e(t, x)$. The Levi-Civita connection pulls back to the covariant derivatives $D_\alpha := \partial_\alpha + iA_\alpha$, which generate curvatures $F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha$. Orthonormality of the frame ensures $A_\alpha \in \mathbb{R}$. The zero-torsion property of the connection enforces the compatibility condition $D_\alpha \psi_\beta = D_\beta \psi_\alpha$. Using the fact that \mathbb{S}^2 has constant curvature $+1$, one may calculate directly that $F_{\alpha\beta} = \text{Im}(\bar{\psi}_\beta \psi_\alpha)$. Similarly, using the fact that \mathbb{H}^2 has constant curvature -1 leads to $F_{\alpha\beta} = -\text{Im}(\bar{\psi}_\beta \psi_\alpha)$. So that we can consider both cases simultaneously we will write $F_{\alpha\beta} = \mu \text{Im}(\bar{\psi}_\beta \psi_\alpha)$, taking $\mu = +1$ for the sphere and $\mu = -1$ for the hyperbolic plane. Thus for any map ϕ and any choice of frame $e(t, x)$, it holds that

$$F_{\alpha\beta} = \mu \text{Im}(\bar{\psi}_\beta \psi_\alpha) \quad \text{and} \quad D_\alpha \psi_\beta = D_\beta \psi_\alpha$$

These relations are all preserved by the transformations

$$\phi \mapsto e^{-i\theta} \phi \quad A \mapsto A + d\theta \quad (5)$$

where $\theta(t, x)$ is a compactly supported real-valued function (we only use time-independent functions in the case of (6)). This gauge invariance corresponds to the freedom we have in the choice of frame $e(t, x)$.

Here and throughout we use ∂_0 and ∂_t interchangeably. We also adopt the convention that Greek indices are allowed to assume values from the set $\{0, 1, 2\}$, whereas Roman indices are restricted $\{1, 2\}$, meaning that Roman indices indicate only spatial variables.

At the gauge field level, the energy-critical harmonic maps equation (2) assumes the form

$$\begin{cases} 0 & = D_j \psi_j \\ F_{12} & = \mu \text{Im}(\bar{\psi}_2 \psi_1) \\ D_1 \psi_2 & = D_2 \psi_1 \end{cases} \quad (6)$$

The procedure for obtaining gauge field representations of evolution equations is slightly less straightforward. For the harmonic map heat flow, for instance, we begin by pulling back the left and right hand sides of equation (3):

$$\psi_t = D_j \psi_j \quad (7)$$

To obtain an evolution equation from (7), we covariantly differentiate in a spatial direction by applying D_k and then invoke the compatibility condition $D_k \psi_t = D_t \psi_k$:

$$D_t \psi_k = D_k D_j \psi_j$$

By using the curvature relation to commute D_k and D_j and then applying the compatibility condition $D_j \psi_k = D_k \psi_j$, we obtain a covariant heat equation for ψ_k . All told, we obtain the system

$$\begin{cases} D_t \psi_k &= D_j D_j \psi_k - i F_{jk} \psi_j \\ F_{01} &= \mu \operatorname{Im}(\bar{\psi}_1 D_j \psi_j) \\ F_{02} &= \mu \operatorname{Im}(\bar{\psi}_2 D_j \psi_j) \\ F_{12} &= \mu \operatorname{Im}(\bar{\psi}_2 \psi_1) \\ D_1 \psi_2 &= D_2 \psi_1 \end{cases} \quad (8)$$

Note that we have eliminated the field ψ_t . The gauge field equations for Schrödinger maps are similarly derived. For (4), the analogue of (7) is

$$\psi_t = i D_j \psi_j$$

and we arrive at the system

$$\begin{cases} D_t \psi_k &= i D_j D_j \psi_k + F_{jk} \psi_j \\ F_{01} &= \mu \operatorname{Re}(\bar{\psi}_1 D_j \psi_j) \\ F_{02} &= \mu \operatorname{Re}(\bar{\psi}_2 D_j \psi_j) \\ F_{12} &= \mu \operatorname{Im}(\bar{\psi}_2 \psi_1) \\ D_1 \psi_2 &= D_2 \psi_1 \end{cases} \quad (9)$$

Remark 1.1. All three of the above systems, i.e., (6), (8), and (9), are preserved by gauge transformations (5). In order to obtain well-defined flows, one must eliminate the gauge freedom by making a gauge choice. See [44, Chapter 6] for a survey of various gauge choices. It appears that the best gauge for handling arbitrary Schrödinger maps (e.g., maps without any symmetry assumption) is the caloric gauge, which was introduced in [43] in the context of wave maps and first applied to Schrödinger maps in [3]. The preferred gauge for studying harmonic maps and Schrödinger maps with equivariant symmetry is the Coulomb gauge.

Remark 1.2. It is natural to ask whether solutions of (6), (8), or (9) must arise from an underlying map. Assuming sufficient decay and regularity, this is indeed the case; see [43] for a discussion in the setting of wave maps.

1.3. Topology. Because $d^2A = 0$ for any 1-form A , the following always holds:

$$\partial_t F_{12} - \partial_1 F_{02} + \partial_2 F_{01} = 0 \quad (10)$$

This simple observation is very useful in studying the gauge field systems (6)—(9).

Definition 1.3. The *first Chern number* c_1 of a vector bundle over \mathbb{R}^2 with connection A is the integral

$$c_1 := \frac{1}{2\pi} \int_{\mathbb{R}^2} d\underline{A} = \frac{1}{2\pi} \int_{\mathbb{R}^2} F_{12} dx^1 \wedge dx^2$$

For solutions (ψ, A) of a gauge field system, we also refer to c_1 as the *charge*.

The “underline” notation introduced here we will also use in the sequel: an underlined form means take only the spatial components of that form. For instance, if $A = A_0 dt + A_j dx^j$, then $\underline{A} = A_j dx^j$.

Lemma 1.4. *The curvature condition (10) implies the conservation law*

$$\partial_t \frac{1}{2\pi} \int_{\mathbb{R}^2} F_{12} dx = 0$$

for rapidly decaying solutions of (8) or (9), i.e., charge is conserved.

A less obvious fact is that for the system (6), charge is *quantized*, which is to say that it is integer-valued. At the level of maps, this follows from the Gauss-Bonnet theorem and the fact that $d\underline{A}$ is the pullback by the map of the volume form on the target. Charge in fact characterizes the homotopy class. To prove quantization at the gauge field level, one exhausts \mathbb{R}^2 with nested discs, applies Stokes’ theorem to the integral of $d\underline{A}$ over each disc, and then controls the resulting integrals over the boundary that arise. The field equations are of course essential in establishing quantization. See [30, Chapter 3] for further discussion.

2. LAGRANGIAN FORMULATION

In this section we show that the system (9) arises as the Euler-Lagrange equations of a suitable gauge-invariant action. The difficulties encountered at the level of the map do not arise. In view of Remark (1.2), this furnishes a Lagrangian formulation for the Schrödinger map system.

In carrying out variations we work formally, assuming smoothness of all quantities and assuming that fields and variations are rapidly decaying.

Theorem 2.1. *The energy-critical gauged Schrödinger map system (9) is generated by the action*

$$L_{Sch}(\psi, A) := \int_{\mathbb{R}^{2+1}} [\operatorname{Re}(\bar{\psi}_2 D_t \psi_1) - \operatorname{Im}(\overline{D_j \psi_2} D_j \psi_1)] dx^1 \wedge dx^2 \wedge dt \\ + \frac{1}{2} \int_{\mathbb{R}^{2+1}} (|\psi_1|^2 + |\psi_2|^2) dt \wedge dA + \mu \frac{1}{2} \int_{\mathbb{R}^{2+1}} A \wedge dA$$

provided that the compatibility condition $D_1 \psi_2 = D_2 \psi_1$ holds at the initial time.

Proof. We verify the variation.

Variation of ψ . The variation of ψ_1, ψ_2 respectively give rise to the $D_t \psi_2$ and $D_t \psi_1$ evolutions of (9).

Under the variation $\psi_1 \mapsto \psi_1 + \varepsilon \phi$, the terms linear in ε from

$$\operatorname{Re}(\bar{\psi}_2 D_t \psi_1), \quad -\operatorname{Im}(\overline{D_j \psi_2} D_j \psi_1), \quad \frac{1}{2} F_{12} |\psi_1|^2,$$

are, respectively,

$$\operatorname{Re}(\bar{\psi}_2 D_t \phi), \quad -\operatorname{Im}(\overline{D_j \psi_2} D_j \phi), \quad F_{12} \operatorname{Re}(\bar{\psi}_1 \phi)$$

Integrating by parts in

$$\int_{\mathbb{R}^{2+1}} [\operatorname{Re}(\bar{\psi}_2 D_t \phi) - \operatorname{Im}(\overline{D_j \psi_2} D_j \phi) + F_{12} \operatorname{Re}(\bar{\psi}_1 \phi)] dx dt$$

yields

$$\int_{\mathbb{R}^{2+1}} [-\operatorname{Re}(\bar{\phi} D_t \psi_2) - \operatorname{Im}(\bar{\phi} D_j D_j \psi_2) + F_{12} \operatorname{Re}(\bar{\phi} \psi_1)] dx dt$$

which leads to the evolution equation

$$D_t \psi_2 = i D_j D_j \psi_2 + F_{12} \psi_1$$

Similarly, under the variation $\psi_2 \mapsto \psi_2 + \varepsilon \phi$ we obtain the ε -linear terms

$$\operatorname{Re}(\bar{\phi} D_t \psi_1), \quad -\operatorname{Im}(\overline{D_j \phi} D_j \psi_1), \quad F_{12} \operatorname{Re}(\bar{\phi} \psi_2)$$

which lead to the evolution equation

$$D_t \psi_1 = i D_j D_j \psi_1 - F_{12} \psi_2$$

Variation of A . The variation of A_0 leads to the F_{12} curvature equation. Varying A_1 and A_2 respectively yield preliminary F_{02} and F_{01} equations. To obtain the compatibility condition $D_1 \psi_2 = D_2 \psi_1$, we enforce it at time zero and then show using Gronwall's inequality that the condition persists. Once we have the compatibility condition, we can substitute it back into the preliminary F_{0j} equations to obtain the equations appearing in (9).

Under the variation $A \rightarrow A + \varepsilon B$, we get from $\mu \frac{1}{2} \int A \wedge dA$ the ε -linear term

$$\mu B \wedge dA$$

which can be verified using Stokes and the fact that for 1-forms A, B we have $d(A \wedge B) = dA \wedge B - A \wedge dB$. Upon expansion, the term appears as

$$\mu \int_{\mathbb{R}^{2+1}} (B_t F_{12} - B_1 F_{02} + B_2 F_{01}) dx dt \quad (11)$$

From $\text{Re}(\bar{\psi}_2 D_t \psi_1)$, we get from the variation of A the ε -linear term

$$- B_t \text{Im}(\bar{\psi}_2 \psi_1) \quad (12)$$

As there are no other A_t variation terms, we conclude from (11) and (12) that

$$F_{12} = \mu \text{Im}(\bar{\psi}_2 \psi_1) \quad (13)$$

We also have ε terms coming from the variation of the A_j . In particular, $-\text{Im}(\overline{D_j \psi_2} D_j \psi_1)$ contributes

$$B_j \text{Re}(\bar{\psi}_2 D_j \psi_1) - B_j \text{Re}(\overline{D_j \psi_2} \psi_1) \quad (14)$$

Finally, we have to handle

$$\frac{1}{2} \int_{\mathbb{R}^{2+1}} (|\psi_1|^2 + |\psi_2|^2) dt \wedge dA$$

To do so we first invoke Stokes to obtain

$$\frac{1}{2} \int_{\mathbb{R}^{2+1}} d[(|\psi_1|^2 + |\psi_2|^2) dt] \wedge A$$

and then expand to get

$$\frac{1}{2} \int_{\mathbb{R}^{2+1}} (A_2 \partial_1 |\psi_x|^2 dx^1 \wedge dt \wedge dx^2 + A_1 \partial_2 |\psi_x|^2) dx^2 \wedge dt \wedge dx^1 \quad (15)$$

Varying (15) with respect to A and then expanding yields the ε -linear terms

$$\int_{\mathbb{R}^{2+1}} [B_1 \text{Re}(\bar{\psi}_1 D_2 \psi_1) + B_1 \text{Re}(\bar{\psi}_2 D_2 \psi_2) - B_2 \text{Re}(\bar{\psi}_1 D_1 \psi_1) - B_2 \text{Re}(\bar{\psi}_2 D_1 \psi_2)] dx dt \quad (16)$$

Comparing the B_1 terms in (11), (14), and (16) leads to

$$\int [\text{Re}(\bar{\psi}_2 D_1 \psi_1) - \text{Re}(\overline{D_1 \psi_2} \psi_1) + \text{Re}(\bar{\psi}_1 D_2 \psi_1) + \text{Re}(\bar{\psi}_2 D_2 \psi_2) - \mu F_{02}] = 0$$

This yields

$$\mu F_{02} = \text{Re}(\bar{\psi}_2 D_j \psi_j) + \text{Re}(\bar{\psi}_1 (D_2 \psi_1 - D_1 \psi_2)) \quad (17)$$

Similarly, comparing B_2 terms leads to

$$\int [\text{Re}(\bar{\psi}_2 D_2 \psi_1) - \text{Re}(\overline{D_2 \psi_2} \psi_1) - \text{Re}(\bar{\psi}_1 D_1 \psi_1) - \text{Re}(\bar{\psi}_2 D_1 \psi_2) + \mu F_{01}] = 0$$

and

$$\mu F_{01} = \text{Re}(\bar{\psi}_1 D_j \psi_j) + \text{Re}(\bar{\psi}_2 (D_1 \psi_2 - D_2 \psi_1)) \quad (18)$$

By direct calculation one may verify that (10) holds with (13), (17), and (18).

The compatibility condition

Set

$$\Theta := D_1\psi_2 - D_2\psi_1$$

Then

$$D_t\Theta = D_1D_t\psi_2 - D_2D_t\psi_1 + iF_{01}\psi_2 - iF_{02}\psi_1 \quad (19)$$

By direct calculation,

$$D_t\psi_1 = iD_1D_j\psi_j - iD_2\Theta$$

$$D_t\psi_2 = iD_2D_j\psi_j + iD_1\Theta$$

which, upon substituting into (19), yield

$$\begin{aligned} D_t\Theta &= i(D_1D_2 - D_2D_1)D_j\psi_j + iD_jD_j\Theta + iF_{01}\psi_2 - iF_{02}\psi_1 \\ &= -F_{12}D_j\psi_j + iF_{01}\psi_2 - iF_{02}\psi_1 + iD_jD_j\Theta \end{aligned}$$

Invoking (13), (17), and (18), we find

$$-F_{12}D_j\psi_j + iF_{01}\psi_2 - iF_{02}\psi_1 = \mu i [\psi_2 \operatorname{Re}(\bar{\psi}_2\Theta) + \psi_1 \operatorname{Re}(\bar{\psi}_1\Theta)]$$

Therefore

$$D_t\Theta = iD_jD_j\Theta + \mu i [\psi_2 \operatorname{Re}(\bar{\psi}_2\Theta) + \psi_1 \operatorname{Re}(\bar{\psi}_1\Theta)]$$

so that in particular

$$\operatorname{Re}(\bar{\Theta}D_t\Theta) = \partial_j \operatorname{Re}(\bar{\Theta}iD_j\Theta) - \mu \operatorname{Im}(\bar{\Theta} [\psi_2 \operatorname{Re}(\bar{\psi}_2\Theta) + \psi_1 \operatorname{Re}(\bar{\psi}_1\Theta)])$$

Consequently

$$\partial_t \frac{1}{2} \int_{\mathbb{R}^2} |\Theta|^2 dx \leq \sup_{\mathbb{R}^2} (|\psi_1|^2 + |\psi_2|^2) \int_{\mathbb{R}^2} |\Theta|^2 dx$$

Therefore if $\Theta = 0$ at time zero, then we conclude by Gronwall's inequality that Θ is zero for all later times for which the solution exists. By time reversibility of the system, this means that the compatibility condition

$$D_1\psi_2 = D_2\psi_1 \quad (20)$$

holds for all times on the interval of existence provided that it holds at at least one point in the interval.

By using the compatibility condition (20) in (18) and (17), we recover the F_{0j} equations of (9). \square

Remark 2.2. The initial data of (ψ, A) may be chosen in any way that is consistent with the curvature constraints and compatibility condition.

Remark 2.3. The time compatibility conditions $D_0\psi_k = D_k\psi_0$ are not present because we have no need for—and therefore have not introduced—the derivative field ψ_0 .

Remark 2.4. An essential a priori difference between our action and that introduced in [32] is that instead of a term quartic in ψ , we introduce a term quadratic in ψ and linear in dA . This has the effect of coupling ψ and dA in such a way so as to yield the desired equations for F_{0j} under no additional assumptions (save that the compatibility condition be satisfied

at the initial time). A posteriori, however, i.e., after the derivation of the Euler-Lagrange equations, these terms are seen to be equivalent.

3. GRADIENT FLOW

Theorem 3.1. *The energy-critical gauged harmonic map heat flow system (8) is generated by the gradient flow of*

$$H_{Har}(\psi, A) := \frac{1}{2} \int_{\mathbb{R}^2} \left(\operatorname{Re}(\overline{D_j \psi_k} D_j \psi_k) - \mu \frac{1}{2} [\operatorname{Im}(\bar{\psi}_2 \psi_1)]^2 \right) dx^1 \wedge dx^2 \\ - \mu \frac{1}{2} \int_{\mathbb{R}^2} d\mathbf{A} \wedge *d\mathbf{A} \quad (21)$$

provided

$$F_{12} = \mu \operatorname{Im}(\bar{\psi}_2 \psi_1), \quad D_1 \psi_2 = D_2 \psi_1 \quad (22)$$

at the initial time.

Proof. We first obtain (8) with $A_0 = 0$. A posteriori one may incorporate $A_0 \neq 0$ in the flow and retain gauge invariance. Note that (21) itself is invariant under time-independent gauge transformations.

Varying ψ_1 leads to

$$- \int_{\mathbb{R}^2} [\operatorname{Re}(\bar{\phi} D_j D_j \psi_1) + \mu \operatorname{Im}(\bar{\psi}_2 \psi_1) \operatorname{Im}(\bar{\psi}_2 \phi)] dx^1 \wedge dx^2$$

The associated downward gradient flow for ψ_1 is therefore

$$\partial_t \psi_1 = D_j D_j \psi_1 + i\mu \operatorname{Im}(\bar{\psi}_2 \psi_1) \psi_2 \quad (23)$$

Similarly, varying ψ_2 leads to

$$\partial_t \psi_2 = D_j D_j \psi_2 - i\mu \operatorname{Im}(\bar{\psi}_2 \psi_1) \psi_1 \quad (24)$$

Varying A_j leads to

$$B_j \operatorname{Im}(\bar{\psi}_k D_j \psi_k) - B_j \mu \partial_k F_{jk}$$

Which gradient direction we choose for A_j depends upon μ : when we couple the F_{0j} equations with (10), we want to obtain a forward heat evolution for F_{12} rather than a backward heat flow. Therefore we take

$$F_{0j} = \partial_t A_j = \mu \operatorname{Im}(\bar{\psi}_k D_j \psi_k) - \partial_k F_{jk} \quad (25)$$

so that coupling (25) with (10) yields

$$(\partial_t - \Delta) F_{12} = \mu [\partial_1 \operatorname{Im}(\bar{\psi}_k D_2 \psi_k) - \partial_2 \operatorname{Im}(\bar{\psi}_k D_1 \psi_k)] \\ = \mu [-2 \operatorname{Im}(\overline{D_2 \psi_k} D_1 \psi_k) + \operatorname{Im}(\bar{\psi}_k (D_1 D_2 - D_2 D_1) \psi_k)] \\ = \mu [-2 \operatorname{Im}(\overline{D_2 \psi_k} D_1 \psi_k) + F_{12} (|\psi_1|^2 + |\psi_2|^2)] \quad (26)$$

On the other hand, using (23), (24), we get

$$\begin{aligned}\partial_t \operatorname{Im}(\bar{\psi}_2 \psi_1) &= \operatorname{Im}(\bar{\psi}_2 \partial_t \psi_1) - \operatorname{Im}(\bar{\psi}_1 \partial_t \psi_2) \\ &= \operatorname{Im}(\bar{\psi}_2 D_j D_j \psi_1) - \operatorname{Im}(\bar{\psi}_1 D_j D_j \psi_2) + \mu \operatorname{Im}(\bar{\psi}_2 \psi_1)(|\psi_1|^2 + |\psi_2|^2) \\ &= \Delta \operatorname{Im}(\bar{\psi}_2 \psi_1) - 2 \operatorname{Im}(\overline{D_j \psi_2} D_j \psi_1) + \mu \operatorname{Im}(\bar{\psi}_2 \psi_1)(|\psi_1|^2 + |\psi_2|^2)\end{aligned}$$

so that

$$(\partial_t - \Delta) \mu \operatorname{Im}(\bar{\psi}_2 \psi_1) = -2 \mu \operatorname{Im}(\overline{D_j \psi_2} D_j \psi_1) + \operatorname{Im}(\bar{\psi}_2 \psi_1)(|\psi_1|^2 + |\psi_2|^2) \quad (27)$$

Comparing (26) and (27) suggests $F_{12} = \mu \operatorname{Im}(\bar{\psi}_2 \psi_1)$ and $D_1 \psi_2 = D_2 \psi_1$, and we enforce this at the initial time. Using a Gronwall inequality argument similar to that in Theorem 2.1, one enforces the compatibility condition at all later times; then one concludes $F_{12} = \mu \operatorname{Im}(\bar{\psi}_2 \psi_1)$ for all time by uniqueness of solutions to the heat equation.

Returning to (25) and having established that conditions (22) hold for the entire interval of existence, we obtain

$$F_{0j} = \mu \operatorname{Im}(\bar{\psi}_j D_k \psi_k)$$

□

Remark 3.2. Harmonic map heat flow is the key tool used to construct the caloric gauge. See [43, 41].

4. SOLITONS

We define solitons as steady states of the above equations. Evident from the geometric map formulation of the evolution equations is that harmonic maps constitute the solitons for harmonic map heat flow and Schrödinger maps.

Lemma 4.1. *If (ψ, A) are smooth and satisfy*

$$\begin{cases} (D_1 + iD_2)\psi_1 &= 0 \\ (D_1 + iD_2)\psi_2 &= 0 \\ F_{12} &= \mu \operatorname{Im}(\bar{\psi}_2 \psi_1) \\ D_1 \psi_2 &= D_2 \psi_1 \end{cases} \quad (28)$$

or

$$\begin{cases} (D_1 - iD_2)\psi_1 &= 0 \\ (D_1 - iD_2)\psi_2 &= 0 \\ F_{12} &= \mu \operatorname{Im}(\bar{\psi}_2 \psi_1) \\ D_1 \psi_2 &= D_2 \psi_1 \end{cases} \quad (29)$$

and $F_{12} \neq 0$ at at least one point, then (ψ, A) satisfy the energy-critical gauged harmonic map system (6) and provide stationary solutions of the gauged harmonic map heat flow system (8) and the gauged Schrödinger map system (9).

Proof. We consider solutions of (28), as solutions of (29) may be handled analogously. Invoking the compatibility condition $D_1\psi_2 = D_2\psi_1$, we conclude from the first two equations of (28) that

$$\begin{cases} D_1(\psi_1 + i\psi_2) &= 0 \\ D_2(\psi_1 + i\psi_2) &= 0 \end{cases} \quad (30)$$

Differentiating (30) leads to

$$(D_1D_2 - D_2D_1)(\psi_1 + i\psi_2) = iF_{12}(\psi_1 + i\psi_2) = 0$$

which in view of the fact that $F_{12} \neq 0$ at at least one point, implies that

$$\psi_1 = -i\psi_2 \quad (31)$$

holds at some $x \in \mathbb{R}^2$. Thanks to (30), we have for $j = 1, 2$ that

$$\partial_j |\psi_1 + i\psi_2|^2 = 0$$

for all $x \in \mathbb{R}^2$. Therefore (31) also holds for all $x \in \mathbb{R}^2$.

Together (31) and (28) imply

$$D_j\psi_j = (D_1 + iD_2)\psi_1 = 0$$

To see that (8) and (9) are satisfied, it only remains to use the compatibility and spatial curvature conditions to write

$$D_jD_j\psi_k - iF_{jk}\psi_j = D_jD_k\psi_j - iF_{jk} = D_kD_j\psi_j$$

□

Remark 4.2. Note that solitons satisfying (28) have

$$\mu F_{12} = -\frac{1}{2}(|\psi_1|^2 + |\psi_2|^2) \leq 0$$

and that solitons satisfying (29) have

$$\mu F_{12} = \frac{1}{2}(|\psi_1|^2 + |\psi_2|^2) \geq 0$$

which in particular fixes the sign of the topological degree. When $F_{12} = \pm \frac{1}{2}(|\psi_1|^2 + |\psi_2|^2)$, topological quantization automatically implies quantization of the quantity

$$\frac{1}{2\pi} \cdot \frac{1}{2} \int_{\mathbb{R}^2} (|\psi_1|^2 + |\psi_2|^2) dx$$

which on the level of maps corresponds to energy-quantization of harmonic maps.

Remark 4.3. Regularity results and the reverse direction of Lemma 4.1 are of interest but we do not establish either here. Also we note that it is not always the case that nontrivial smooth finite-charge solutions exist.

5. CONSERVATION LAWS

Conservation laws are developed at the gauge level in [32] and at the level of maps in [18]. Our approach here is at the gauge level, in the spirit of [1, 7].

We begin by introducing the pseudo-stress-energy tensor $T_{\alpha\beta}$, defined via

$$\begin{cases} T_{00} &= \frac{1}{2}(|\psi_1|^2 + |\psi_2|^2) \\ T_{0j} &= \text{Im}(\bar{\psi}_\ell D_j \psi_\ell) \\ T_{jk} &= 2\text{Re}(\overline{D_j \psi_\ell} D_k \psi_\ell) - \delta_{jk} \Delta T_{00} \end{cases} \quad (32)$$

Theorem 5.1. *Solutions (ψ, A) of the energy-critical gauged Schrödinger map system (9) satisfy the conservation law*

$$\partial_\alpha T_{0\alpha} = 0 \quad (33)$$

and the balance law

$$\partial_\alpha T_{j\alpha} = 2F_{\alpha j} T_{0\alpha} \quad (34)$$

Proof. First we establish (33). Using the evolution equations in (9), we have

$$\begin{aligned} \frac{1}{2} \partial_t |\psi_1|^2 &= \text{Re}(\bar{\psi}_1 D_t \psi_1) \\ &= \text{Re}(\bar{\psi}_1 i D_j D_j \psi_1) + \text{Re}(\bar{\psi}_1 F_{j1} \psi_j) \\ &= \partial_j \text{Re}(\bar{\psi}_1 i D_j \psi_1) + F_{21} \text{Re}(\bar{\psi}_1 \psi_2) \end{aligned}$$

and

$$\frac{1}{2} \partial_t |\psi_2|^2 = \partial_j \text{Re}(\bar{\psi}_2 i D_j \psi_2) + F_{12} \text{Re}(\bar{\psi}_2 \psi_1)$$

Consequently,

$$\frac{1}{2} \partial_t (|\psi_1|^2 + |\psi_2|^2) = \partial_j \text{Re}(\bar{\psi}_\ell i D_j \psi_\ell)$$

Next we show (34), which is more involved. We start by using the evolution and curvature conditions to obtain

$$\begin{aligned} \partial_t T_{0j} &= \text{Im}(\overline{D_t \psi_\ell} D_j \psi_\ell) + \text{Im}(\bar{\psi}_\ell D_t D_j \psi_\ell) \\ &= \text{Im}(\overline{i D_k D_k \psi_\ell} D_j \psi_\ell) + \text{Im}(\overline{F_{k\ell} \psi_k} D_j \psi_\ell) \\ &\quad + \text{Im}(\bar{\psi}_\ell D_j D_t \psi_\ell) + \text{Im}(\bar{\psi}_\ell i F_{0j} \psi_\ell) \end{aligned} \quad (35)$$

The rightmost term of (35) can be rewritten as

$$\text{Im}(\bar{\psi}_\ell i F_{0j} \psi_\ell) = F_{0j} (|\psi_1|^2 + |\psi_2|^2) = 2F_{0j} T_{00}$$

In view of the evolution equation, curvature conditions, and compatibility condition, the first term of the last line of (35) expands as

$$\begin{aligned}
\operatorname{Im}(\bar{\psi}_\ell D_j D_t \psi_\ell) &= \operatorname{Im}(\bar{\psi}_\ell i D_j D_k D_k \psi_\ell) + \operatorname{Im}(\bar{\psi}_\ell D_j (F_{k\ell} \psi_k)) \\
&= \operatorname{Im}(\bar{\psi}_\ell i D_k D_j D_k \psi_\ell) - \operatorname{Im}(\bar{\psi}_\ell F_{jk} D_k \psi_\ell) + \operatorname{Im}(\bar{\psi}_\ell D_j (F_{k\ell} \psi_k)) \\
&= \operatorname{Im}(\bar{\psi}_\ell i D_k D_j D_k \psi_\ell) + \operatorname{Im}(\bar{\psi}_\ell D_j (F_{k\ell} \psi_k)) + F_{kj} T_{0k} \\
&= \partial_k \operatorname{Im}(\bar{\psi}_\ell i D_j D_k \psi_\ell) - \operatorname{Im}(\overline{D_k \psi_\ell} i D_j D_k \psi_\ell) \\
&\quad + \operatorname{Im}(\bar{\psi}_\ell D_j (F_{k\ell} \psi_k)) + F_{kj} T_{0k}
\end{aligned}$$

Appealing only to the curvature conditions and compatibility condition, we rewrite the first term of the second to last line of (35) as

$$\begin{aligned}
\operatorname{Im}(\overline{i D_k D_k \psi_\ell} D_j \psi_\ell) &= \partial_k \operatorname{Im}(\overline{i D_k \psi_\ell} D_j \psi_\ell) - \operatorname{Im}(\overline{i D_k \psi_\ell} D_k D_j \psi_\ell) \\
&= \partial_k \operatorname{Im}(\overline{i D_\ell \psi_k} D_j \psi_\ell) - \operatorname{Im}(\overline{i D_\ell \psi_k} D_k D_j \psi_\ell)
\end{aligned}$$

where we rewrite $-\operatorname{Im}(\overline{i D_\ell \psi_k} D_k D_j \psi_\ell)$ as

$$\begin{aligned}
-\operatorname{Im}(\overline{i D_\ell \psi_k} D_k D_j \psi_\ell) &= -\operatorname{Im}(\overline{i D_\ell \psi_k} D_j D_k \psi_\ell) - \operatorname{Im}(\overline{i D_\ell \psi_k} i F_{kj} \psi_\ell) \\
&= -\operatorname{Im}(\overline{i D_\ell \psi_k} D_j D_k \psi_\ell) - F_{kj} \operatorname{Im}(\overline{D_k \psi_\ell} \psi_\ell) \\
&= -\operatorname{Im}(\overline{i D_\ell \psi_k} D_j D_k \psi_\ell) + F_{kj} T_{0k}
\end{aligned}$$

so that

$$\operatorname{Im}(\overline{i D_k D_k \psi_\ell} D_j \psi_\ell) = \partial_k \operatorname{Im}(\overline{i D_\ell \psi_k} D_j \psi_\ell) - \operatorname{Im}(\overline{i D_\ell \psi_k} D_j D_k \psi_\ell) + F_{kj} T_{0k}$$

Together

$$\begin{aligned}
&\operatorname{Im}(\bar{\psi}_\ell D_j D_t \psi_\ell) + \operatorname{Im}(\overline{i D_k D_k \psi_\ell} D_j \psi_\ell) \\
&= \partial_k \operatorname{Im}(\bar{\psi}_\ell i D_j D_k \psi_\ell) + \partial_k \operatorname{Im}(\overline{i D_\ell \psi_k} D_j \psi_\ell) + \operatorname{Im}(\bar{\psi}_\ell D_j (F_{k\ell} \psi_k)) + 2F_{kj} T_{0k}
\end{aligned}$$

Therefore

$$\begin{aligned}
\partial_t T_{0j} &= 2F_{\alpha j} T_{0\alpha} \\
&\quad + \partial_k \operatorname{Im}(\bar{\psi}_\ell i D_j D_k \psi_\ell) + \partial_k \operatorname{Im}(\overline{i D_\ell \psi_k} D_j \psi_\ell) \\
&\quad + \operatorname{Im}(\bar{\psi}_\ell D_j (F_{k\ell} \psi_k)) + \operatorname{Im}(\overline{F_{k\ell} \psi_k} D_j \psi_\ell)
\end{aligned} \tag{36}$$

The last two lines of (36) may be rewritten as

$$\begin{aligned}
&\partial_k \partial_j \operatorname{Im}(\bar{\psi}_\ell i D_k \psi_\ell) - 2\partial_k \operatorname{Im}(\overline{D_j \psi_\ell} i D_k \psi_\ell) \\
&\quad + \partial_j \operatorname{Im}(\bar{\psi}_\ell F_{k\ell} \psi_k) - 2\operatorname{Im}(\overline{D_j \psi_\ell} F_{k\ell} \psi_k)
\end{aligned} \tag{37}$$

The first term of (37) can be rewritten as $\partial_j \Delta T_{00}$. The third term of (37) is $\mu \partial_j F_{12}^2$. For the fourth term, we have

$$-2\operatorname{Im}(\overline{D_j \psi_\ell} F_{k\ell} \psi_k) = -2F_{12} \mu \partial_j F_{12} = -\mu \partial_j F_{12}^2$$

Therefore

$$\partial_t T_{0j} = -2\partial_k \operatorname{Im}(\overline{D_j \psi_\ell} i D_k \psi_\ell) + \partial_j \Delta T_{00} + 2F_{\alpha j} T_{0\alpha}$$

□

Corollary 5.2. *For rapidly decaying solutions of (9), the quantity*

$$\frac{1}{2} \int_{\mathbb{R}^2} (|\psi_1|^2 + |\psi_2|^2) dx$$

is conserved.

Corollary 5.3. *Define the 1-form T via*

$$T = T_{00}dt + T_{0j}dx^j$$

Then the conservation law (33) implies

$$d^*T = 0$$

from which we conclude that there exists a 2-form potential U satisfying

$$d^*U = T$$

Lemma 5.4. *Let*

$$H_{Sch} := \int_{\mathbb{R}^2} \left(-\text{Im}(\overline{D_j \psi_2} D_j \psi_1) + \frac{1}{2}(|\psi_1|^2 + |\psi_2|^2) F_{12} \right) dx^1 \wedge dx^2 \quad (38)$$

Then, for rapidly decaying solutions of the gauged Schrödinger map system (9), it holds that

$$H_{Sch} = \frac{1}{2} \int_{\mathbb{R}^2} d\underline{T} = \frac{1}{2} \int_{\mathbb{R}^2} (\partial_1 T_{02} - \partial_2 T_{01}) dx^1 \wedge dx^2 = 0$$

Proof. Using the compatibility and curvature conditions, we calculate

$$\begin{aligned} \text{Im}(\overline{D_1 \psi_2} D_1 \psi_1) &= \text{Im}(\overline{D_2 \psi_1} D_1 \psi_1) \\ &= \partial_1 \text{Im}(\overline{D_2 \psi_1} \psi_1) - \text{Im}(\overline{D_1 D_2 \psi_1} \psi_1) \\ &= \partial_1 \text{Im}(\overline{D_2 \psi_1} \psi_1) - \text{Im}(\overline{D_2 D_1 \psi_1} \psi_1) - \text{Im}(\overline{i F_{12} \psi_1} \psi_1) \end{aligned} \quad (39)$$

The right hand side we may expand as

$$\partial_1 \text{Im}(\overline{D_2 \psi_1} \psi_1) - \partial_2 \text{Im}(\overline{D_1 \psi_1} \psi_1) + \text{Im}(\overline{D_1 \psi_1} D_2 \psi_1) + F_{12} |\psi_1|^2$$

which by virtue of the string of equalities in (39) is equal to $\text{Im}(\overline{D_2 \psi_1} D_1 \psi_1)$.

This implies

$$2\text{Im}(\overline{D_2 \psi_1} D_1 \psi_1) = \partial_1 \text{Im}(\overline{D_2 \psi_1} \psi_1) - \partial_2 \text{Im}(\overline{D_1 \psi_1} \psi_1) + F_{12} |\psi_1|^2$$

and hence

$$\text{Im}(\overline{D_1 \psi_2} D_1 \psi_1) = \frac{1}{2} (\partial_1 \text{Im}(\overline{D_2 \psi_1} \psi_1) - \partial_2 \text{Im}(\overline{D_1 \psi_1} \psi_1) + F_{12} |\psi_1|^2)$$

By conjugating and reversing the roles of the indices, we similarly conclude

$$\text{Im}(\overline{D_2 \psi_2} D_1 \psi_1) = \frac{1}{2} (\partial_2 \text{Im}(\overline{\psi_2} D_1 \psi_2) - \partial_1 \text{Im}(\overline{\psi_2} D_2 \psi_2) + F_{12} |\psi_2|^2)$$

Therefore

$$-\text{Im}(\overline{D_j \psi_2} D_j \psi_1) + \frac{1}{2}(|\psi_1|^2 + |\psi_2|^2) F_{12} = \frac{1}{2} (\partial_1 \text{Im}(\overline{\psi_2} D_2 \psi_j) - \partial_2 \text{Im}(\overline{\psi_j} D_1 \psi_j))$$

□

6. VIRIAL IDENTITIES AND MORAWETZ ESTIMATES

We follow the presentation in [7], though we refer the reader also to [8, 39].

A virial identity was established in the context of equivariant Schrödinger maps in [1]. For virial and Morawetz identities in the context of radial Schrödinger maps, see [19]. Following [39], both [40] and [12] establish some frequency-localized estimates in the setting of Schrödinger maps by using virial and Morawetz-type arguments.

Define the virial potential by

$$V_a(t) = \int_{\mathbb{R}^2} a(x) T_{00} dx$$

and the Morawetz action by

$$M_a(t) = \int_{\mathbb{R}^2} T_{0j} \partial_j a \, dx$$

The conservation law (33) followed by integration by parts implies

$$\partial_t V_a(t) = M_a(t)$$

We recover the generalized virial identity from [7, Lemma 3.1], adapted to the setting of Schrödinger maps.

Lemma 6.1. *Let $a : \mathbb{R}^2 \rightarrow \mathbb{R}$ and let (ψ, A) be a solution of (9). Then*

$$M_a(T) - M_a(0) = \int_0^T \int_{\mathbb{R}^2} [2\operatorname{Re}(\overline{D_j \psi_\ell} D_k \psi_\ell) \partial_j \partial_k a - T_{00} \Delta^2 a + 2F_{\alpha j} T_{0\alpha} \partial_j a] \, dx dt$$

Proof. Using the Morawetz action, balance law, and integration by parts, we have

$$\partial_t M_a(t) = 2 \int_{\mathbb{R}^2} (T_{jk} \partial_k \partial_j a + F_{\alpha j} T_{0\alpha} \partial_j a) \, dx$$

□

We also have its corollary:

Corollary 6.2. *If a is convex, then we can further conclude that*

$$\int_0^T \int_{\mathbb{R}^2} (2F_{\alpha j} T_{0\alpha} \partial_j a - T_{00} \Delta^2 a) \, dx dt \lesssim \sup_{[0, T]} |M_a(t)|$$

Interaction Morawetz estimates

Recalling $T_{\alpha\beta}$ defined in (32), let

$$\rho = T_{00}, \quad p_j = T_{0j}$$

and

$$T_{jk} = \sigma_{jk} - \delta_{jk} \Delta \rho$$

where

$$\sigma_{jk} = 2\operatorname{Re}(\overline{D_j\psi_\ell} D_k\psi_\ell)$$

We can then rewrite the conservation law (33) as

$$\partial_t \rho + \partial_j p_j = 0$$

and the balance law (34) as

$$\partial_t p_j + \partial_k (\sigma_{jk} - \delta_{jk} \Delta \rho) = 2(F_{0j} \rho + F_{kj} p_k)$$

Take a tensor product of two solutions $(\psi^{(1)}, A^{(1)})$ and $(\psi^{(2)}, A^{(2)})$. The corresponding Morawetz action is

$$M(t) = \int_{\mathbb{R}^2 \otimes \mathbb{R}^2} T_{j0} \partial_j a \, dx dy$$

Set $a = |x - y|$. It follows by direct calculation that

$$M(t) = \int_{\mathbb{R}^2 \otimes \mathbb{R}^2} T_{j0} \partial_j a \, dx dy = 4 \int_{\mathbb{R}^2 \otimes \mathbb{R}^2} \frac{x_j - y_j}{|x - y|} p_j(t, x) \rho(t, y) dx dy$$

The dimension of the underlying space suggests following the commutator vector operator approach introduced in [8]. As indicated by the remarks in [7, §4], it remains to suitably control the balance terms $F_{\alpha j} T_{0\alpha}$.

7. COMPARISON WITH CHERN-SIMONS-SCHRÖDINGER

In two spatial dimensions, the Chern-Simons-Schrödinger equation arises as the second-quantization of a nonrelativistic anyon system. For background, see [26, 10, 45, 16, 17, 23, 24, 25, 31]. Local wellposedness at high regularity is established in [5] using the Coulomb gauge and at low-regularity for small data in [29] using the heat gauge. In the setting of Chern-Simons-Schrödinger systems, the heat gauge appears to have been first introduced in [9]. The small-data critical wellposedness problem is open.

Lagrangian formulation

The action is

$$L(\phi, A) = \frac{1}{2} \int_{\mathbb{R}^{2+1}} \left[\operatorname{Im}(\bar{\phi} D_t \phi) + |D_x \phi|^2 - \frac{g}{2} |\phi|^4 \right] dx^1 \wedge dx^2 \wedge dt + \frac{1}{2} \int_{\mathbb{R}^{2+1}} A \wedge dA$$

with Euler-Lagrange equations

$$\begin{cases} D_t \phi &= i D_\ell D_\ell \phi + i g |\phi|^2 \phi \\ F_{01} &= -\operatorname{Im}(\bar{\phi} D_2 \phi) \\ F_{02} &= \operatorname{Im}(\bar{\phi} D_1 \phi) \\ F_{12} &= -\frac{1}{2} |\phi|^2 \end{cases} \quad (40)$$

both of which enjoy the gauge freedom (5). It is interesting to note that (40) is Galilean invariant, whereas (9) is not; the obstruction lies with the compatibility condition. In the above g is a coupling constant. The so-called “critical coupling” is $g = \frac{1}{2}$, and this is what we consider below.

Gradient flow

The gradient flow of

$$\frac{1}{2} \int_{\mathbb{R}^2} \left[|D_x \phi|^2 - \frac{1}{4} |\phi|^4 \right] dx^1 \wedge dx^2 + \frac{1}{2} \int_{\mathbb{R}^2} d\underline{A} \wedge *d\underline{A}$$

yields

$$\begin{cases} D_t \phi &= D_\ell D_\ell \phi + \frac{1}{2} |\phi|^2 \phi \\ F_{01} &= -\text{Im}(\bar{\phi} D_1 \phi) - \partial_2 F_{12} \\ F_{02} &= -\text{Im}(\bar{\phi} D_2 \phi) + \partial_1 F_{12} \end{cases}$$

which with $d^2 A = 0$ imply

$$(\partial_t - \Delta) F_{12} = \partial_1 \text{Im}(\bar{\phi} D_2 \phi) - \partial_2 \text{Im}(\bar{\phi} D_1 \phi)$$

In analogy with the caloric gauge for Schrödinger maps, the Chern-Simons gradient flow can be used to construct a caloric gauge for the Chern-Simons-Schrödinger system. Both caloric gauges can be interpreted as modifications of the Coulomb gauge; in the case of Chern-Simons-Schrödinger, the modification is not sufficient to render the caloric gauge a favorable alternative to the Coulomb gauge for the purposes of establishing wellposedness. The source of the difference lies in the F_{12} curvature, which in the case of Schrödinger maps exhibits a null-form-like cancellation, but in the case of Chern-Simons-Schrödinger behaves like $|\phi|^2$. The author thanks Daniel Tataru for sharing this observation.

Solitons

Stationary solutions of (40) are the so-called self-dual and anti-self-dual Chern-Simons solitons. These are solutions of

$$\begin{cases} (D_1 + iD_2)\phi &= 0 \\ F_{01} &= -\text{Im}(\bar{\phi} D_2 \phi) \\ F_{02} &= \text{Im}(\bar{\phi} D_1 \phi) \\ F_{12} &= -\frac{1}{2} |\phi|^2 \end{cases}$$

or, respectively,

$$\begin{cases} (D_1 - iD_2)\phi &= 0 \\ F_{01} &= -\text{Im}(\bar{\phi} D_2 \phi) \\ F_{02} &= \text{Im}(\bar{\phi} D_1 \phi) \\ F_{12} &= -\frac{1}{2} |\phi|^2 \end{cases}$$

In this case A_0 is not identically zero so long as ϕ is not. We refer the reader to [14, 13].

Conservation laws

For the Chern-Simons-Schrödinger system (40), we set, following [7],

$$\begin{cases} T_{00} &= \frac{1}{2}|\phi|^2 \\ T_{0j} &= \text{Im}(\bar{\phi}D_j\phi) \\ T_{jk} &= 2\text{Re}(\overline{D_j\phi}D_k\phi) - \delta_{jk}(T_{00} + \Delta)T_{00} \end{cases}$$

Here there is no distinction between the conservation law (33) and the curvature relation (10). Note that for ϕ not identically zero we always have

$$\int_{\mathbb{R}^2} dA = - \int_{\mathbb{R}^2} T_{00} dx < 0$$

The balance law (34) is still valid in this context:

$$\partial_\alpha T_{j\alpha} = 2F_{\alpha j}T_{0\alpha}$$

The right hand side, however, vanishes thanks to $F_{01} = -T_{02}$, $F_{02} = T_{01}$, and $F_{12} = -T_{00}$, so that

$$\partial_\alpha T_{j\alpha} = 0$$

A caveat, however, is that T_{jk} in this setting incorporates a $-\delta_{jk}T_{00}^2$ term.

Lemma 7.1. *For sufficiently regular solutions (ϕ, A) of (40), the quantity*

$$E(\phi) := \frac{1}{2} \int_{\mathbb{R}^2} \left(|D_x\phi|^2 - \frac{1}{4}|\phi|^4 \right) dx$$

is conserved.

Proof. The result may be verified by direct calculation using the system (40). Note that

$$E(\phi) = \frac{1}{4} \int_{\mathbb{R}^2} (T_{11} + T_{22}) dx$$

□

Virial identities

The $-\delta_{jk}T_{00}^2$ term in T_{jk} adds a term in the generalized virial identity with a sign that is unfavorable for establishing Morawetz estimates.

Lemma 7.2. *Let $a : \mathbb{R}^2 \rightarrow \mathbb{R}$ and let (ϕ, A) be a solution of (40). Then the Morawetz action*

$$M_a(t) = \int_{\mathbb{R}^2} T_{0j} \partial_j a \, dx$$

satisfies

$$M_a(T) - M_a(0) = \int_0^T \int_{\mathbb{R}^2} [2\text{Re}(\overline{D_j\phi}D_k\phi)\partial_j\partial_k a - T_{00}\Delta^2 a - T_{00}^2\Delta a] \, dx dt$$

For $a = |x|^2$, it holds that

$$\partial_t^2 \int_{\mathbb{R}^2} |x|^2 T_{00} dx = \partial_t M_{\{a=|x|^2\}}(t) = 2 \int_{\mathbb{R}^2} (|D_x \phi|^2 - T_{00}^2) dx = 4E(\phi) \quad (41)$$

Equation (41) was used in [5] to establish the existence of finite-time blow-up solutions by taking data with negative energy or data with positive energy and sufficiently large weighted momentum. We remark that [22] constructs finite-time blow-up solutions that have zero energy. The fact that (41) holds is closely tied with exact conservation laws and pseudo-conformal invariance [44, §2.4]. Solutions in [22] were constructed by exploiting pseudo-conformal invariance. In the case of Schrödinger maps, (34) is not an exact conservation law. Moreover, pseudo-conformal invariance fails to hold, the obstruction being the compatibility condition [21]. If the compatibility condition were dropped, then H_{Sch} introduced in (38) could be made to be nonzero, but not otherwise. Constructing blow-up solutions for Schrödinger maps is therefore more involved [34, 33, 38]; see also the complementary stability result [4].

REFERENCES

- [1] I. Bejenaru, A. Ionescu, C. E. Kenig, and D. Tataru, *Equivariant Schrödinger maps in two spatial dimensions*, ArXiv e-prints: 1112.6122 (2011).
- [2] ———, *Equivariant Schrödinger Maps in two spatial dimensions: the \mathbb{H}^2 target*, ArXiv e-prints: 1212.2566 (2012).
- [3] I. Bejenaru, A. D. Ionescu, C. E. Kenig, and D. Tataru, *Global Schrödinger maps in dimensions $d \geq 2$: small data in the critical Sobolev spaces*, Ann. of Math. (2) **173** (2011), no. 3, 1443–1506. MR 2800718
- [4] I. Bejenaru and D. Tataru, *Near soliton evolution for equivariant Schroedinger Maps in two spatial dimensions*, ArXiv e-prints: 1009.1608 (2010).
- [5] L. Bergé, A. De Bouard, and J.-C. Saut, *Blowing up time-dependent solutions of the planar, Chern-Simons gauged nonlinear Schrödinger equation*, Nonlinearity **8** (1995), no. 2, 235–253. MR 1328596 (96b:81025)
- [6] Nai-Heng Chang, Jalal Shatah, and Karen Uhlenbeck, *Schrödinger maps*, Comm. Pure Appl. Math. **53** (2000), no. 5, 590–602. MR 1737504 (2001a:35161)
- [7] J. Colliander, M. Czubak, and J. Lee, *Interaction Morawetz estimate for the magnetic Schrödinger equation and applications*, ArXiv e-prints: 1112.4452 (2011).
- [8] J. Colliander, M. Grillakis, and N. Tzirakis, *Tensor products and correlation estimates with applications to nonlinear Schrödinger equations*, Comm. Pure Appl. Math. **62** (2009), no. 7, 920–968. MR 2527809 (2010c:35175)
- [9] Sophia Demoulini, *Global existence for a nonlinear Schroedinger-Chern-Simons system on a surface*, Ann. Inst. H. Poincaré Anal. Non Linéaire **24** (2007), no. 2, 207–225. MR 2310693 (2008k:35439)
- [10] S. Deser, R. Jackiw, and S. Templeton, *Topologically massive gauge theories*, Ann. Physics **140** (1982), no. 2, 372–411. MR 665601 (84j:81128)
- [11] Weyue Ding and Youde Wang, *Local Schrödinger flow into Kähler manifolds*, Sci. China Ser. A **44** (2001), no. 11, 1446–1464. MR 1877231 (2002k:53129)
- [12] B. Dodson, *Bilinear Strichartz estimates for the Schrödinger map problem*, ArXiv e-prints: 1210.5255 (2012).
- [13] Gerald Dunne, R. Jackiw, So-Young Pi, and Carlo A. Trugenberger, *Erratum: “Self-dual Chern-Simons solitons and two-dimensional nonlinear equations [Phys. Rev. D*

- (3) **43** (1991), no. 4, 1332–1345; MR1104647 (92g:81120)], Phys. Rev. D (3) **45** (1992), no. 8, 3012. MR 1157129
- [14] Gerald V. Dunne, R. Jackiw, So-Young Pi, and Carlo A. Trugenberger, *Self-dual Chern-Simons solitons and two-dimensional nonlinear equations*, Phys. Rev. D (3) **43** (1991), no. 4, 1332–1345. MR 1104647 (92g:81120)
- [15] James Eells, Jr. and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160. MR 0164306 (29 #1603)
- [16] Z. F. Ezawa, M. Hotta, and A. Iwazaki, *Breathing vortex solitons in nonrelativistic Chern-Simons gauge theory*, Phys. Rev. Lett. **67** (1991), no. 4, 411–414. MR 1114940 (92f:81100)
- [17] ———, *Nonrelativistic Chern-Simons vortex solitons in external magnetic field*, Phys. Rev. **44** (1991), no. D, 452–63.
- [18] Manoussos G. Grillakis and Vagelis Stefanopoulos, *Lagrangian formulation, energy estimates, and the Schrödinger map problem*, Comm. Partial Differential Equations **27** (2002), no. 9–10, 1845–1877. MR 1941660 (2003i:58042)
- [19] S. Gustafson and E. Koo, *Global well-posedness for 2D radial Schrödinger maps into the sphere*, ArXiv e-prints: 1105.5659 (2011).
- [20] Frédéric Hélein, *Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne*, C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), no. 8, 591–596. MR 1101039 (92e:58055)
- [21] Hyungjin Huh, *Blow-up solutions of modified Schrödinger maps*, Comm. Partial Differential Equations **33** (2008), no. 1–3, 235–243. MR 2398227 (2009i:35301)
- [22] ———, *Blow-up solutions of the Chern-Simons-Schrödinger equations*, Nonlinearity **22** (2009), no. 5, 967–974. MR 2501032 (2010d:35340)
- [23] R. Jackiw and So-Young Pi, *Time-dependent Chern-Simons solitons and their quantization*, Phys. Rev. D (3) **44** (1991), no. 8, 2524–2532. MR 1132645 (93d:81108)
- [24] ———, *Self-dual Chern-Simons solitons*, Progr. Theoret. Phys. Suppl. (1992), no. 107, 1–40, Low-dimensional field theories and condensed matter physics (Kyoto, 1991). MR 1194691 (94d:81114)
- [25] R. Jackiw, So-Young Pi, and Erick J. Weinberg, *Topological and nontopological solitons in relativistic and nonrelativistic Chern-Simons theory*, Particles, strings and cosmology (Boston, MA, 1990), World Sci. Publ., River Edge, NJ, 1991, pp. 573–588. MR 1170608
- [26] R. Jackiw and S. Templeton, *How super-renormalizable interactions cure their infrared divergences*, Phys. Rev. **23** (1993), no. B, 2291–304.
- [27] L. D. Landau, *Collected papers of L. D. Landau*, Edited and with an introduction by D. ter Haar. Second printing, Gordon and Breach Science Publishers, New York, 1967. MR 0237287 (38 #5577)
- [28] Fanghua Lin and Changyou Wang, *The analysis of harmonic maps and their heat flows*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008. MR 2431658 (2011a:58030)
- [29] B. Liu, P. Smith, and D. Tataru, *Local wellposedness of Chern-Simons-Schrödinger*, ArXiv e-prints: 1212.1476 (2012).
- [30] Nicholas Manton and Paul Sutcliffe, *Topological solitons*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge, 2004. MR 2068924 (2006d:58020)
- [31] L. Martina, O. K. Pashaev, and G. Soliani, *Self-dual Chern-Simons solitons in nonlinear σ -models*, Modern Phys. Lett. A **8** (1993), no. 34, 3241–3250. MR 1249658 (94g:81138)
- [32] ———, *Quantization of planar ferromagnets in the Chern-Simons representation*, Teoret. Mat. Fiz. **99** (1994), no. 3, 450–461. MR 1308811 (95j:81150)
- [33] F. Merle, P. Raphaël, and I. Rodnianski, *Blow up dynamics for smooth equivariant solutions to the energy critical Schrödinger map*, ArXiv e-prints: 1106.0912 (2011).

- [34] Frank Merle, Pierre Raphaël, and Igor Rodnianski, *Blow up dynamics for smooth equivariant solutions to the energy critical Schrödinger map*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 5-6, 279–283. MR 2783320 (2012e:58029)
- [35] Andrea Nahmod, Jalal Shatah, Luis Vega, and Chongchun Zeng, *Schrödinger maps and their associated frame systems*, Int. Math. Res. Not. IMRN (2007), no. 21, Art. ID rnm088, 29. MR 2352219 (2009b:58063)
- [36] Andrea Nahmod, Atanas Stefanov, and Karen Uhlenbeck, *On Schrödinger maps*, Comm. Pure Appl. Math. **56** (2003), no. 1, 114–151. MR 1929444 (2003i:58030)
- [37] N. Papanicolaou and T. N. Tomaras, *Dynamics of magnetic vortices*, Nuclear Phys. B **360** (1991), no. 2-3, 425–462. MR 1118793 (92c:82114)
- [38] G. Perelman, *Blow up dynamics for equivariant critical Schrödinger maps*, ArXiv e-prints: 1212.6709 (2012).
- [39] Fabrice Planchon and Luis Vega, *Bilinear virial identities and applications*, Ann. Sci. Éc. Norm. Supér. (4) **42** (2009), no. 2, 261–290. MR 2518079 (2010b:35441)
- [40] Paul Smith, *Conditional global regularity of Schroedinger maps: sub-threshold dispersed energy*, to appear in Analysis & PDE.
- [41] ———, *Geometric renormalization below the ground state*, Int. Math. Res. Not. IMRN (2012), no. 16, 3800–3844. MR 2959028
- [42] Michael Struwe, *On the evolution of harmonic mappings of Riemannian surfaces*, Comment. Math. Helv. **60** (1985), no. 4, 558–581. MR 826871 (87e:58056)
- [43] Terence Tao, *Geometric renormalization of large energy wave maps*, Journées “Équations aux Dérivées Partielles”, École Polytech., Palaiseau, 2004, pp. Exp. No. XI, 32. MR 2135366 (2006i:58044)
- [44] ———, *Nonlinear dispersive equations*, CBMS Regional Conference Series in Mathematics, vol. 106, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2006, Local and global analysis. MR 2233925 (2008i:35211)
- [45] Frank Wilczek, *Fractional statistics and anyon superconductivity*, World Scientific Publishing Co. Inc., Teaneck, NJ, 1990. MR 1081990 (92c:81001)

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